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# ON THE PROBLEM OF DYNAMIC CONTACT ANGLE*

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Plane motion of a viscous incompressible fluid bounded by a rectangular rigid wall and a free boundary of constant form is investigated. The free boundary is in contact with the rigid wall at a point which moves along the wall, coming into contact with it at a constant rate. The asymptotics of the velocity field near the point of contact is computed under the assumption that the motion is stationary in the coordinate system attached to the moving free boundary and, that the energy is dissipated as a finite rate.

To date a fairly detailed study has been made of the mathematical formulation of problems concerning the steady state motion of a viscous fluid in the presence of points or lines of contact between the free boundary and the solid wall (**).

It is essential that the point (or line) of contact is assumed fixed with respect to the wall at rest. This makes possible the formulation of the boundary condition for the approach of the free boundary to the point of contact at a prescribed angle determined by the condition of static equilibrium of the three media at this point /5/ (see also the discussion of this problem in /6/). The situation becomes much more complicated when the point of contact is in motion. The difficulties arising will be explained using an example of stationary motion in a coordinate system attached to the point of contact. Such a motion can be realized, e.g. in the problem of filling a capillary /7/. The formulation of a plane analog of the capillary filling problem in given in Sect.2, and the flow is discussed below is assumed to be plane. The assumption is not essential, but reduces the amount of manipulation.

**1. Basic hypothesis.** We consider a stationary solution of the Navier–Stokes equations in the region $\Omega \subset R^2$ bounded by a free boundary $\Gamma$ described by the equation $x_2 = f(x_1)$ and a rigid wall $x_2 = 0$ moving along the $x_1$-axis with velocity of $-V \neq 0$ (Fig.1). We assume that $f(0) = 0$, $f > 0$ for $|x_1| < a$ with certain $a > 0$, and that $f \in C^2[-0, a]$. We introduce the following notation: $R_+^2$ denotes the half-plane $x_2 > 0$, $\Pi_a$ is a semicircle $x_1^2 + x_2^2 < a^2$, $x_2 > 0$, $\theta_m$ is the angle between the tangent ot $\Gamma$ at the point of contact $O$ and the negative direction of the $x_1$-axis and $\mathbf{n}$ is the unit vector of the outward normal to curve $\Gamma$. Since $\Gamma \in C^2$, it follows that $\mathbf{n}$ is a smooth vector field. A smooth continuation of the field $\mathbf{n}$ into the region $\Omega \cap \Pi_a = \omega$ exists at sufficiently smooth $a$, and we shall continue to denote it by $\mathbf{n}$.

We assume that the components $v_1$ and $v_2$ of the velocity vector $\mathbf{v}$ belong to the Sobolev space $W_2^1(\omega)$ (the assumption guarantees the finiteness of velocity of the dissipation rate and the kinetic energy of the fluid near the point $O$). Let us define the functions

$$v_n = \mathbf{v}\cdot\mathbf{n}, \quad w(x_1, x_2) = \begin{cases} \zeta(|x|)\, v_n(x), & x \in \omega \\ 0, & x \in \Pi_a \setminus \omega \end{cases}$$

Here $\zeta(s)$ is a smooth shear function equal to unity for $0 \leqslant s \leqslant a/2$ and to zero for $s \geqslant a$. By virtue of the kinematic condition we have $v_{n \setminus \Gamma} = 0$ at the free boundary. This, together with the inclusion $v_i \in W_2^1(\omega)$, $i = 1, 2$ implies that $w \in W_2^1(R_+^2)$. Let us obtain an estimate from below for the integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|w(x_1, 0) - w(y_1, 0)|^2}{|x_1 - y_1|^2}\, dx_1\, dy_1$$

Since $w(x_1, 0) = 0$ when $x_1 \geqslant 0$, we have

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**) See /1–4/ and also Solonnikov V.A. "Solvability of the problem of plane flow of a viscous incompressible capillary fluid in an open vessel". Preprint LOMI, Leningrad, No.P-5-77; and Jean M. "Free surface of the stationary flow of a Newtonian fluid in a finite channel", Preprint de la Laboratoire de mécanique et d'acoustique, Centre national de la recherche Scientifique. Marseille, 1979.

$$I \geqslant \int_{-\infty}^{0}\left(\int_{0}^{\infty}\frac{dy_1}{|x-y_1|^2}\right)|w(x_1,0)|^2\,dx_1 = -\int_{-\infty}^{0}\frac{|w(x_1,0)|^2}{x_1}\,dx_1$$

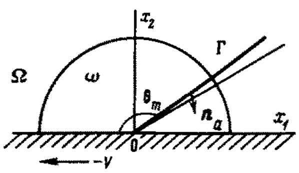


Fig.1

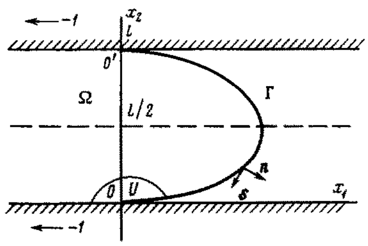


Fig.2

By virtue of the adhesion condition, we have $w(x_1, 0) \to -V\sin\theta_m$ as $x_1 \to -0$. If $\theta_m \neq 0, \theta_m \neq \pi$, then $w(-0, 0) \neq 0$ and the integral in the right-hand part of the inequality for $I$ diverges logarithmically as $x_1 \to -0$. On the other hand, using the known trace theorem /8/ we find that for any function $w \in W_2^1(R_+^2)$,

$$I \leqslant C\int_{x_2>0}|\nabla w|^2\,dx$$

where $C > 0$ is a certain constant. It follows therefore that when $0 < \theta_m < \pi$, the Dirichlet integral of the function $w$ is infinite. This in turn contradicts the assumption that $v_i \in W_2^1(\omega)$, $i = 1, 2$.

We note that we made no use in the above discussion of the fact that the functions $v_1$ and $v_2$ together with the corresponding pressure $p$, satisfy the Navier–Stokes equations. The contradiction arose from the incompatibility of the adhesion conditions, conditions at the free boundary, and the assumption $\theta_m \neq 0$, $\theta_m \neq \pi$ in the case of a moving point of contact. The incompatability was first noted in /9/ (see also /5/, a remark in p.350). Various methods of removing this contradiction are available. In /10–12/ the finite values for the energy dissipation and force of friction against the wall were obtained by replacing, at some section of the wall, near the point of contact the adhesion condition, by various forms of the condition of slippage. In /7,13/ an asymptotic solution was given for the problem of a fluid moving along a plane wall, the solution containing a single empirical parameter, namely the angle $\alpha$ of inclination of the free boundary at the distance $h$ from the wall. Here $h$ is a small quantity of the order of several intermolecular distances.

The absence of a correct mathematical formulation for the problem of a moving point of contact increases the difficulties encountered in computing the similar flows. Thus when the problem of filling a capillary was solved numerically in /14/, the computations were arbitrarily terminated at some distance from the wall. The computational method given in /5/ ignores the presence of a singularity at the point of contact altogether.

Let us formulate the basic hypothesis of the present paper. We assume that under the conditions listed above (plane stationary motion in a coordinate system attached to the point of contact, and an unwetted solid surface), with the fluid flowing forward, the angle $\theta_m$ at which the free boundary approaches the wall, in equal to $\pi$ (Fig.2). If in addition the free boundary is a Liapunov curve up to the point of contact and the velocity field belongs to some Hölder weight class (Sect.5), then we construct an asymptotics of the flow in the neighborhood of the point of contact. The asymptotics is such, that the tangential stress at the wall has an integrable singularity, the rate of energy dissipation near the point 0 is finite, and the asymptotics of the free boundary as $x_1 \to +0$ is $x_2 = O(x_1^{2-\varkappa})$ where $\varkappa = \pi^{-1}\operatorname{arctg} 2k$, $0 < \varkappa < 1/2$ (exact formulation is given in Sect.6). Here $k$ is a dimensionless parameter (we shall call it the capillary number) equal to $\rho\nu V/\sigma$ where $\rho$ is the density of the fluid, $\nu$ is the kinematic viscosity coefficient and $\sigma$ is the coefficient of surface tension.

It can be expected that the proposed hypothesis corresponds to a real flow of a poorly wetting fluid along a dry surface of low roughness. The above assumption is reinforced by the fact that at large values of the static wetting angle $\theta_s$ the observed dynamic angle (identified with $\theta_m$) is also large. The closer it is to $\pi$, the larger the parameter $k = \rho\nu V/\sigma$ (see /16/ where a large amount of experimental data was analyzed and /6/, which gives a survey of the experiments and models for the problem of dynamic contact angle). We also note the approximate solution of the problem of dynamic contact angle given in /7/ for the angles of inclination of the free boundary close to $\pi$, loses its validity when the capillary number $k \gg 1$, but it is precisely in this case, as /16/ implies, that the angle $\theta_m$ is almost equal to $\pi$.

## 2. Formulation of the problem.

Let us formulate a plane problem of symmetrical filling of a capillary. We require to find a smooth curve $\Gamma$ passing through the points $x_1 = 0$, $x_2 = 0$ and $x_1 = 0$, $x_2 = l$, and a solution $\mathbf{v}$, $p$ of the Navier–Stokes equations

$$-\Delta\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} + \nabla p = 0, \quad \operatorname{div}\mathbf{v} = 0 \tag{2.1}$$

in the region $\Omega$ bounded by two semiinfinite straight lines $x_1 < 0, x_2 = 0$ and $x_1 < 0, x_2 = l$, and by the curve $\Gamma$ (Fig.2), with the following boundary conditions:

$$v_1 = -1,\ v_2 = 0,\ x_1 < 0,\ x_2 = 0 \text{ and } x_2 = l \tag{2.2}$$

$$\mathbf{v}\cdot\mathbf{n} = 0,\ \mathbf{s}S\mathbf{n} = 0,\quad K = k\,(p - 2\mathbf{n}S\mathbf{n}),\quad (x_1, x_2) \in \Gamma \tag{2.3}$$

Here $S$ is the deformation rate tensor, $K$ is the curvature of the free boundary $\Gamma$, $\mathbf{s}$ and $\mathbf{n}$ are the unit vectors of the tangent and normal to $\Gamma$. We assume that $K > 0$ if $\Gamma$ is convex away from the fluid. The vectors $\mathbf{s}$ and $\mathbf{n}$ define the right orientation of the $x_1x_2$-plane.

The equations (2.1) – (2.3) are written in dimensionless variables. The quantities $\nu/V$, $V$ and $\rho V^2$ are chosen as the characteristic dimensions of length, velocity and pressure respectively ($V$ is the physical velocity of displacement of the free boundary along the capillary). We assume that there are no external forces acting on the fluid. The function $p$ denotes the difference between the pressure in the fluid and the atmospheric pressure, the latter assumed constant. The parameter $k$ in the condition (2.3) represents the capillary No. defined in Sect.1. The equations (2.2) express the conditions of adhesion of the fluid to the moving walls of the capillary, $x_2 = 0$ and $x_2 = l$. The first condition of (2.3) on the free boundary means that the curve $\Gamma$ is a streamline. According to the second condition the tangential stress at $\Gamma$ is zero. The third condition of (2.3) expresses the fact that the normal stress at the free boundary is equal to the capillary pressure /5/. We impose on the solution sought, in addition to (2.2) and (2.3), the condition of symmetry about the axis of the capillary $x_2 = l/2$. We require that the curve $\Gamma$ be symmetrical about this line and, that after the substitution $x_1 - l/2 = \bar{x}_1$ the function $v_1$ and $p$ become even, while $v_2$ becomes an odd function of $\bar{x}_1$.

Since the region $\Omega$ is not compact, the solution of the system (2.1) will require a certain condition to be formulated for $x_1 \to -\infty$. It is natural to assume that the motion realized at a distance from the free boundary is close to the superposition of the Poiseuille flow and a uniform flow. In other words,

$$\begin{gathered} v_1 \to -1 + 6l^{-2}x_1\,(l - x_1),\quad v_2 \to 0 \\ \partial p/\partial x_1 \to -12l^{-2},\ \partial p/\partial x_2 \to 0,\ x_1 \to -\infty \end{gathered} \tag{2.4}$$

The limiting pressure gradient in (2.4) is such, that the rate of flow of fluid across a section of the capillary is zero, and this agrees with the first condition of (2.3). To close the formulation of the problem, we must specify a condition at the points of contact $O$ and $O'$. In accordance with the hypothesis of Sect.1, we shall assume that the curve $\Gamma$ touches the lines $x_2 = 0$, $x_2 = l$ at the points $O$ and $O'$ respectively (Fig.2).

There are grounds for assuming that the problem (2.1)– (2.4) with additional symmetry and tangency conditions is formulated correctly, although so far no proof has been obtained. Below we study the asymptotic behavior of the solution of the problem in question near the point of contact $O$, under the assumption that a solution exists. We note that the proposed method of investigating the asymptotic behavior is fully applicable to the case of an axisymmetric flow in a circular capillary. Moreover, the resulting expressions for the free boundary (7.1) and tangential stress at the wall (7.2) have the same form when written for a circular capillary in the local coordinates attached to the point of contact.

In what follows, it will be conveneint to write the relations (2.1)– (2.3) in terms of the stream function $\psi$ connected with the velocity components by the relations $v_1 = \partial\psi/\partial x_2$, $v_2 = -\partial\psi/\partial x_1$. According to (2.2), (2.3) the stream function retains its constant value along the whole boundary of the region $\Omega$, and we can assume it to be equal to zero without loss of generality. The equation for the stream function and the boundary conditions (2.2) have the form

$$\Delta\Delta\psi + \frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1} = 0,\quad (x_1, x_2) \in \Omega \tag{2.5}$$

$$\psi = 0,\ \partial\psi/\partial x_2 = -1,\ x_1 < 0,\ x_2 = 0 \text{ and } x_2 = l \tag{2.6}$$

The conditions (2.3) at the free boundary assume the form /17/

$$\psi = 0,\quad \Delta\psi - 2K\frac{\partial\psi}{\partial n} = 0,\quad \frac{\partial\Delta\psi}{\partial n} + 2\frac{\partial^2}{\partial s^2}\left(\frac{\partial\psi}{\partial n}\right) - \frac{1}{2}\frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial n}\right)^2 - k^{-1}\frac{\partial K}{\partial s} = 0,\quad (x_1, x_2) \in \Gamma \tag{2.7}$$

We assume that the field $\mathbf{v}$ is continous at the pint $O$ at which the boundary conditions change. Since we have $v_1 = -1$ at the coordinate origin, the inequality $v_1 < 0$ holds for sufficiently small $|x|$. In this case a subregion $U$ of $\Omega$ exists such, that $O \in \bar{U}$ and $\partial\psi/\partial x_2 \leqslant -1/2$ in $\bar{U}$. This makes possible the passage to the Mises variables.

**3. Mises variables.** Let us pass in the relations (2.5)–(2.7) to new variables $x = x_1, \psi$ and new unknown function $x_2 = y(x, \psi)$. In place of (2.5) we now have

$$y_\psi MMy - My(My)_\psi - y_\psi^{-1}(My)_x = 0 \tag{3.1}$$

$$M\left(y_\alpha, y_\psi, \frac{\partial}{\partial x}, \frac{\partial}{\partial \psi}\right) = \frac{1}{y_\psi}\frac{\partial^2}{\partial x^2} - \frac{2y_x}{y_\psi^2}\frac{\partial^2}{\partial x \partial \psi} + \frac{1+y_x^2}{y_\psi^3}\frac{\partial^2}{\partial \psi^2}$$

Although the line $\Gamma$ is not known in the initial variables, in the Mises variables $x, \psi$ its image will be a segment of the straight line $\psi = 0$. We can assume without loss of generality that the image of $U$ is a semicircle $\Pi_\varepsilon = \{x, \psi : x^2 + \psi^2 < \varepsilon^2, \psi < 0\}$. Moreover, $-2 \leqslant y_\psi \leqslant -1/2$ in $\Pi_\varepsilon$ provided that $\varepsilon > 0$ is sufficiently small. The mapping $(x_1, x_2) \to (x, \psi)$ is one-sheeted in $\Pi_\varepsilon$.

The conditions on the wall (2.6) and at the free boundary (2.7) are written in the Mises variables as follows /17/:

$$y = 0, \quad y_\psi = -1, \quad x < 0, \quad \psi = 0 \tag{3.2}$$

$$y_\psi My - \frac{2y_{xx}}{1+y_x^2} = 0, \quad -\frac{1+y_x^2}{y_\psi}(My)_\psi + \tag{3.3}$$

$$y_x(My)_x + 2\left[\frac{1}{\sqrt{1+y_x^2}}\left(\frac{\sqrt{1+y_x^2}}{y_\psi}\right)_x\right]_x +$$

$$k^{-1}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{xx} - \frac{1}{2}\left(\frac{1+y_x^2}{y_\psi}\right)_x = 0, \quad x > 0, \psi = 0$$

The solutions of the equation (3.1) with conditions (3.2), (3.3) close to $y = -\psi$ are of interest. We note that the function $-\psi$ itself satisfies (3.1)–(3.3) and the corresponding flow in the $x_1x_2$-plane is a uniform stream. We set

$$y = -\psi + z(x, \psi) \tag{3.4}$$

and substitute (3.4) into (3.1)–(3.3). After separating from the resulting equations the terms containing higher order derivatives linear with respect to the new unknown function $z$, we obtain

$$\Delta\Delta z = \varphi_0(z_x, z_\psi, \ldots, z_{\psi\psi\psi\psi}), \quad (x, \psi) \in \Pi_\varepsilon \tag{3.5}$$

$$z = 0, \quad z_\psi = 0, \quad -\varepsilon < x < 0, \quad \psi = 0$$

$$z_{\psi\psi} - z_{xx} = \varphi_2(z_x, z_\psi, \ldots, z_{\psi\psi}), \quad z_{\psi\psi\psi} + 3z_{xx\psi} -$$

$$k^{-1}z_{xxx} = \varphi_1(z_x, z_\psi, \ldots, z_{\psi\psi\psi}), \quad 0 < x < \varepsilon, \quad \psi = 0$$

$$\varphi_0 = (\Delta L + L\Delta - L^2)z + z_\psi(\Delta - L)^2 z - (\Delta - L)z[(\Delta - L)z]_\psi - \tag{3.6}$$

$$(1 - z_\psi)^{-1}[(\Delta - L)z]_x$$

$$\varphi_2 = z_\psi \Delta z + (1 - z_\psi)Lz - 2(1 + z_x^2)^{-1} z_x^2 z_{xx}$$

$$\varphi_1 = (Lz)_\psi - \frac{z_\psi + z_x^2}{1 - z_\psi}[(\Delta - L)z]_\psi - z_x[(\Delta - L)z]_x -$$

$$\left[\frac{z_x z_{xx}}{(1 - z_\psi)(1 + z_x^2)}\right]_x - 2\left[\frac{(2z_\psi - z_\psi^2)z_{x\psi}}{(1 - z_\psi)^2}\right]_x +$$

$$\frac{1}{2}\left(\frac{1 + z_x^2}{1 - z_\psi}\right)_x + k^{-1}\left(z_x\left(\frac{1}{\sqrt{1 + z_x^2}} - 1\right)\right]_{xx},$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \psi^2}, \quad L\left(z_x, z_\psi, \frac{\partial}{\partial x}, \frac{\partial}{\partial \psi}\right) = -\frac{z_\psi}{1 - z_\psi}\frac{\partial^2}{\partial x^2} -$$

$$\frac{2z_x}{(1 - z_\psi)^2}\frac{\partial^2}{\partial x \partial \psi} + \frac{-3z_\psi + z_x^2 + 3z_\psi^2 - z_\psi^3}{(1 - z_\psi)^3}\frac{\partial^2}{\partial \psi^2}$$

The right-hand sides of (3.5) are of second order of smallness with respect to the derivatives of $z$ when they tend to zero. Separating in these equations the principal (at small $z$) terms, we can hope that it is precisely these terms that determine the leading term of the asymptotics of the flow near the point of contact $x = \psi = 0$. We note that by virtue of (3.4) and of the definition of $y$, the equation of the free boundary in the plane of flow is $x_2 = z(x_1, 0)$. Assuming now that $\varphi_0, \varphi_1, \varphi_2$ are known functions of $x$ and $\psi$ defined over the whole half-plane $\psi < 0$ and on the whole $x$-axis respectively, we find that the problem (3.5) (in the conditions of which we must put $\varepsilon = \infty$), can be solved explicitly using the Mellin transform. Sect. 4 deals with its solution.

**4. The formal asymptotics.** We consider the linear problem

$$\Delta\Delta z = \zeta_0(x, \psi), \quad -\infty < x < \infty, \quad \psi < 0 \tag{4.1}$$

$$z = 0, \quad z_\psi = 0, \quad x < 0, \quad \psi = 0$$

$$z_{\psi\psi} - z_{xx} = \zeta_2(x), \quad z_{\psi\psi\psi} + 3z_{xx\psi} - k^{-1}z_{xxx} = \zeta_1(x), \quad x > 0, \quad \psi = 0$$

where $\zeta_0, \zeta_1, \zeta_2$ are known functions of their arguments. We shall assume them to be sufficiently smooth and such, that $\zeta_0 = 0$ when $x^2 + \psi^2 \geqslant 1$, $\zeta_1 = 0$, $\zeta_2 = 0$ and $x \geqslant 1$. Let us pass to the polar coordinates $\rho = (x^2 + \psi^2)^{1/2}$, $\varphi = \operatorname{arctg}(\psi/x)$ in the $x\psi$-plane. We find the solution of the problem (4.1) written in the new variables with help of the Mellin transform

$$g^*(\lambda) = \int_0^\infty g(\rho)\rho^{i\lambda-1}\,d\rho \tag{4.2}$$

Setting

$$z^* = w(\varphi, \lambda), \quad g_j^* = G_j(\varphi, \lambda) \quad (g_j = \rho^{4-j}\zeta_j(\rho\cos\varphi, \ \rho\sin\varphi),\ j = 0, 1, 2)$$

and assuming that $i\lambda = \alpha$, we arrive at the following boundary value problem for the function $w$:

$$\left[\frac{\partial^2}{\partial\varphi^2} + (\alpha+2)^2\right]\left(\frac{\partial^2}{\partial\varphi^2} + \alpha^2\right)w = G_0 \quad -\pi < \varphi < 0 \tag{4.3}$$
$$w = 0,\ w_\varphi = 0,\ \varphi = -\pi$$
$$w_{\varphi\varphi} - \alpha(\alpha+2)w = G_2,\ w_{\varphi\varphi\varphi} + (3\alpha^2 + 6\alpha + 4)w_\varphi + k^{-1}\alpha(\alpha+1)(\alpha+2)w = G_1,\ \ \varphi = 0$$

Let us now explain at which values of $\alpha$ the homogeneous problem (4.3) has nontrivial solutions. If $\alpha \neq 0, \alpha \neq -1, \alpha \neq -2$, the solution of the homogeneous equation for $w$ has the form

$$w = C_1\cos\alpha\varphi + C_2\sin\alpha\varphi + C_3\cos(\alpha+2)\varphi + C_4\sin(\alpha+2)\varphi$$

where $C_1, \ldots, C_4$ are arbitrary constants. Substitution of $w$ into the boundary conditions (4.3) yields a system linear in $C_1, \ldots, C_4$ with the determinant

$$A(\alpha) = 8\alpha(\alpha+1)^2(\alpha+2)\cos\alpha\pi\,(2\cos\alpha\pi - k^{-1}\sin\alpha\pi)$$

Apart from $\alpha = 0, -1, -2$, the following values of $\alpha$ ($m$ is an arbitrary integer) are zeros of the determinant $A(\alpha)$;

$$\alpha = i\lambda = m + {}^1/_2,\ \alpha = i\lambda = m + \varkappa;\ \varkappa = \pi^{-1}\operatorname{arctg}2k$$

Additional computations show that the homogeneous problem (4.3) has no nontrivial solutions when $\alpha = 0, \alpha = -1, \alpha = -2$. Let us write its nontrivial solutions (eigenfunctions) corresponding to the values $\alpha = -{}^3/_2$ and $\alpha = \varkappa - 2$

$$w = w_0 \equiv \sin\varphi/2 + \sin 3\varphi/2,\ \ \alpha = -{}^3/_2 \tag{4.4}$$
$$w = w_1 \equiv (\varkappa - 2)\sin\varkappa\varphi - \varkappa\sin(\varkappa-2)\varphi - 2k(\varkappa-2)\cos\varkappa\varphi + 2k\varkappa\cos(\varkappa-2)\varphi,\ \ \alpha = \varkappa - 2$$

The above solutions of (4.3) have the following corresponding solutions of the homogeneous problem (4.1):

$$z_0 = \rho^{7/2}w_0(\varphi),\ z_1 = \rho^{2-\varkappa}w_1(\varphi)$$

The determinant $A(\alpha)$ has simple zeros at the points $\alpha = m + {}^1/_2, \alpha = m + \varkappa$, therefore the problem (4.3) has no associated functions. The value $\alpha = -{}^3/_2$ is the smallest value of $\alpha$ for which the solution $z = \rho^{-\alpha}w(\varphi, -i\alpha)$ of the homogeneous problem (4.1) and the functions $z_\rho, \rho^{-1}z_\varphi$ all tend to zero as $\rho \to 0$. There are grounds to expect that the asymptotics of the solution $z$ of the nonlinear problem (3.5) satisfying the conditions $z \to 0, \nabla z \to 0$, has the following form as $\rho \to 0$:

$$z = a_0\rho^{7/2}w_0(\varphi) + a_1\rho^{2-\varkappa}w_1(\varphi) + \eta(\rho, \varphi) \tag{4.5}$$

where $a_0$ and $a_1$ are certain constants, $w_0$ and $w_1$ are functions defined in (4.4), and $\eta$ is the remainder term. Indeed, replacing $z$ in (3.5) by the first two terms of (4.5) shows that the order of every terms of the resulting left-hand parts is less, as $\rho \to 0$, than the order of the right-hand parts.

The remainder of the paper provides the substantiation for the representations (4.5), i.e. proves that $\eta = O(\rho^{2-\gamma})$, $|\nabla\eta| = O(\rho^{1-\gamma})$ as $\rho \to 0$, etc. Here $\gamma$ is a some positive number smaller than $\varkappa$ (we recall that $0 < \varkappa < {}^1/_2$ and $\varkappa \to 0$ as $k \to 0, \varkappa \to {}^1/_2$ as $k \to \infty$). Use of the formulas of the form (4.5) in solving the Dirichlet and Neumann problems as well as for solving the linear problems of the theory of elasticity is well known (see e.g. /18/). They were established in /19,20/ with the purpose of solving the general linear elliptic boundary value problems. The author of /21/ obtained the asymptotics in the neighborhood of the corner point of the solution of the first boundary value problem for the quasilinear equation (2.5) equivalent to the Navier-Stokes system.

**5. Functional spaces.** Let us determine the basic functional spaces used in studying the elliptic boundary value problems in the regions with irregular boundaries, and the problems with different boundary conditions at different parts of the boundary (see /19,22/). Let, as in Sect.4, $\Pi_\varepsilon \subset R^2$ be a semicircle $|x| = (x_1^2 + x_2^2)^{1/2} < \varepsilon$, $x_2 < 0$ and let $T_\varepsilon$ denote the segment $(0, \varepsilon)$ of the real axis. Let $n$ and $\mu$ be nonnegative numbers and $n$ be an integer. Under $H_\mu^n(\Pi_\varepsilon)$ we shall understand the space of functions which have, in $\Pi_\varepsilon$, the derivatives of up to the $n$-th order inclusive and the finite norm

$$\|u\|_{H_\mu^n(\Pi_\varepsilon)} = \Big(\sum_{0\leqslant j_1+j_2\leqslant n} \int_{\Pi_\varepsilon} \Big|\frac{\partial^{j_1+j_2}u}{\partial x_1^{j_1}\partial x_2^{j_2}}\Big|^2 |x|^{2\mu-2n+2(j_1+j_2)}dx\Big)^{1/2} \tag{5.1}$$

generalized in the S.L. Sobolev sense. Under $H_\mu^{n+1/2}(T_\varepsilon)$ we shall understand a space of functions defined on $T_\varepsilon$, with the derivatives of up to and including the $n$-th order, and the finite norm

$$\|u\|_{H_\mu^{n+1/2}(\Pi_\varepsilon)} = \Big(\sum_{j=0}^{n} \int_0^\varepsilon \Big|\frac{\partial^j u}{\partial x^j}\Big|^2 |x|^{2\mu-2n-1+2j}\,dx + \tag{5.2}$$

$$\int_0^\varepsilon |x|^{2\mu}\,dx \int_{x/2}^{x} \Big|\frac{\partial^n u(x)}{\partial x^n} - \frac{\partial^n u(y)}{\partial y^n}\Big|^2 \frac{dy}{|x-y|^2}\Big)^{1/2}$$

Finally, when $l > 0$ is not an integer and $s < l$, we define the space $C_s^{0,l}(\Pi_\varepsilon)$ as a set of $[l]$-times continuously differentiable functions with a finite norm ($[l]$ denotes the integral part of $l$)

$$|u|_{C_s^{0,l}(\Pi_\varepsilon)} = \sum_{0\leqslant j_1+j_2\leqslant [l]} \sup |x|^{j_1+j_2-s} \Big|\frac{\partial^{j_1+j_2}u}{\partial x_1^{j_1}\partial x_2^{j_2}}\Big| +$$

$$\sum_{j_1+j_2=[l]} \sup_{x\in\Pi_\varepsilon} |x|^{l-s} \sup_{y\in\Pi_\varepsilon:|x-y|\leqslant|x|/2} |x-y|^{[l]-l} \Big|\frac{\partial^{j_1+j_2}u(x)}{\partial x_1^{j_1}\partial x_2^{j_2}} - \frac{\partial^{j_1+j_2}u(y)}{\partial y_1^{j_1}\partial y_2^{j_2}}\Big|$$

We define the space $C_s^{0,l}(T_\varepsilon)$ in exactly the same manner, and its norm is given by the formula

$$|u|_{C_s^{0,l}(T_\varepsilon)} = \sum_{j=0}^{[l]} \sup_{T_\varepsilon} |x|^{j-s} \Big|\frac{d^j u}{dx^j}\Big| +$$

$$\sup_{x\in T_\varepsilon} |x|^{l-s} \sup_{y\in T_\varepsilon:|x-y|\leqslant|x|/2} |x-y|^{[l]-l} \Big|\frac{d^{[l]}u(x)}{dx^{[l]}} - \frac{d^{[l]}u(y)}{dy^{[l]}}\Big|$$

**Lemma 1.** For $u \in H_\mu^n(\Pi_\varepsilon)$ it is sufficient that

$$\Big|\frac{\partial^{j_1+j_2}u}{\partial x_1^{j_1}\partial x_2^{j_2}}\Big| \leqslant C|x|^{s-j_1-j_2}, \quad s > n-1-\mu, \quad j_1+j_2 \leqslant n+1$$

and for $u \in H_\mu^{n+1/2}(T_\varepsilon)$ it is sufficient that

$$\Big|\frac{d^j u}{dx^j}\Big| \leqslant C|x|^{s-j}, \quad s > n-\mu, \quad j = 0,\ldots,n+1$$

Indeed, under the conditions of the lemma all integrals converge in the norms (5.1) and (5.2), and this can be easily confirmed.

**Lemma 2.** If $u \in H_\mu^n(\Pi_\varepsilon)$ and $n > 1$, then

$$\sup_{\Pi_\varepsilon} |x|^{1+\mu-n}|u(x)| \leqslant C\|u\|_{H_\mu^n(\Pi_\varepsilon)} \tag{5.3}$$

and the constant $C > 0$ is independent of $u$ and $\varepsilon$ when $0 < \varepsilon \leqslant \varepsilon_0$.

**Proof.** Let $x \in \Pi_\varepsilon$ and $K_x = \{y \in \Pi_\varepsilon\colon |x|/2 < |y| < |x|\}$. We use the known multiplicational estimate /23/

$$|u(x)| \leqslant C_1 \Big\{\Big(\sum_{j_1+j_2=n} \Big\|\frac{\partial^{j_1+j_2}u}{\partial x_1^{j_1}\partial x_2^{j_2}}\Big\|_{L_2(K_x)}^2\Big)^{1/2n} \|u\|_{L_2(K_x)}^{(n-1)/n} + |x|^{-1}\|u\|_{L_2(K_x)}\Big\}$$

Let us multiply both parts ofthe inequality by $|x|^{1+\mu-n}$. Since

$$|x|^a \|v\|_{L_2(K_x)} \leqslant C_2(a)\Big(\int_{\Pi_\varepsilon} |v|^2 |y|^{2a}\, dy\Big)^{1/2}$$

then the above inequality yields (5.3) for any real $a$.

**6. Substantiation of the validity of the asymptotic expansion.** First we consider the linear problem (4.1).

**Lemma 3.** Let the function $z(x, \psi) \in H_\mu^4(\Pi_\varepsilon)$ have a finite norm $\sup (x^2+\psi^2)^{-(s+1)/2}|z(x,\psi)|$ $(x, \psi) \in \Pi_\varepsilon$ and satisfy the relations (4.1) in which $\zeta_0 \in C^{0,\alpha}_{s-3}(\Pi_\varepsilon)$, $\zeta_1 \in C^{0,1+\alpha}_{s-2}(T_\varepsilon)$, $\zeta_2 \in C^{0,2+\alpha}_{s-1}(T_\varepsilon)$, with $\alpha, s \in (0, 1)$. Then $z \in C^{0,4+\alpha}_{1+s}(\Pi_{\varepsilon'})$, $\varepsilon' \in (0, \varepsilon)$ and

$$|z|_{C^{0,4+\alpha}_{s-1}(\Pi_{\varepsilon'})} \leqslant C\big(|\zeta_0|_{C^{0,\alpha}_{s-3}(\Pi_\varepsilon)} + |\zeta_1|_{C^{0,\alpha+1}_{s-2}(T_\varepsilon)} + |\zeta_2|_{C^{0,\alpha+2}_{s-1}(T_\varepsilon)} + \sup_{\Pi_\varepsilon}(x^2+\psi^2)^{-(s+1)/2}|z(x,\psi)|\big)$$

The lemma is well known. It follows from the theorems on smoothness of solutions of the elliptic boundary value problems right up to the boundary, and from the local Schauder estimates. The following assertion implied by the results of /19,20/ concerns the behavior of the solution of the problem (4.1) near the coordinate origin, at which point the boundary conditions change.

**Theorem 1.** Let the function $z \in H_\mu^4(\Pi_\varepsilon)$ satisfy the relations (4.1) in the region $\Pi_\varepsilon$, and let $\zeta_0 \in H_{\mu_1}^0(\Pi_\varepsilon)$, $\zeta_1 \in H_{\mu_1}^{1/2}(T_\varepsilon)$, $\zeta_2 \in H_{\mu_1}^{3/2}(T_\varepsilon)$ where $\mu \in (3/2, 2)$ and $\mu_1 < \mu$. If $\mu_1 > 3/2$, then $z \in H_{\mu_1}^4(\Pi_{\varepsilon_1})$ for any $\varepsilon_1 \in (0, \varepsilon)$. If on the other hand $\mu_1 \in (1, 1+\varkappa)$, then the formula (4.5) in which the function $\eta \in H_{\mu_1}^4(\Pi_{\varepsilon_1})$ also satisfies (4.1), holds for $z$.

Theorem 1 provides the justification for the formula (4.5) also for the nonlinear problem (3.5).

**Theorem 2.** Let the function $z \in C^{0,4+\alpha}_{1+\delta}(\Pi_\varepsilon)$ where $\alpha$ and $\delta$ are arbitrarily small positive numbers, satisfy the relations (3.5) in $\Pi_\varepsilon$. Then the formula (4.5) in which $\eta \in C^{0,4+\alpha}_{2-\gamma}(\Pi_{\varepsilon_1})$, with arbitrary $\varepsilon_1 \in (0, \varepsilon)$, $\gamma \in (0, \varkappa)$, holds also for this function.

**Proof.** It can be confirmed that the functions

$$\varphi_0(z_x, \ldots, z_{\psi\psi\psi\psi}) = \Phi_0(x, \psi), \quad \varphi_1(z_x, \ldots, z_{\psi\psi\psi})|_{\psi=0} = \Phi_1(x)$$
$$\varphi_2(z_x, \ldots, z_{\psi\psi})|_{\psi=0} = \Phi_2(x)$$

defined by the formulas (3.6) have the following properties: if $z \in C^{0,4+\alpha}_{1+s}(\Pi_\varepsilon)$, $s \in (0, 1)$, then

$$\Phi_0 \in C^{0,\alpha}_{2s-3}(\Pi_\varepsilon), \quad \Phi_1 \in C^{0,1+\alpha}_{2s-2}(T_\varepsilon), \quad \Phi_2 \in C^{0,2+\alpha}_{2s-1}(T_\varepsilon)$$

and hence by virtue of the lemma 1

$$\Phi_0 \in H_\mu^0(\Pi_\varepsilon),\ \Phi_1 \in H_\mu^{1/2}(T_\varepsilon),\ \Phi_2 \in H_\mu^{3/2}(T_\varepsilon) \tag{6.1}$$

for any $\mu \in (2-2s, 2)$. It follows that under the conditions of the inclusion theorem (6.1) holds for any $\mu = \mu_0$ from the intervals $(\max(3/2, 2-2\delta), 2)$. Consequently by virtue of the theorem we have 1 $z \in H_{\mu_0}^4(\Pi_{\varepsilon_2})$, $\forall \varepsilon_2 < \varepsilon$. By virtue of Lemma 2 the value $\sup \rho^{-(s_1+1)}|z(x,\psi)|$, $(x,\psi) \in \Pi_{\varepsilon_2}$ where $\rho = (x^2+\psi^2)^{1/2}$, is bounded for all $s_1$ of the form $s_1 = 2 - \mu_0 \in (0, \min(1/2, 2\delta))$, and from Lemma 3 we can conclude that $z \in C^{0,4+\alpha}_{1+s_1}(\Pi_{\varepsilon_2})$. The number $s_1$ can be larger than $\delta$. Repeating the above arguments we can, if necessary, show in the end that $z \in C^{0,4+\alpha}_{1+s}(\Pi_{\varepsilon_3})$ with any $\varepsilon_3 < \varepsilon$, $s < 1/2$. Let us take the value of $s$ sufficiently close to $1/2 : s \in ((1-\varkappa)/2, 1/2)$. Then

$$z \in H_\mu^4(\Pi_{\varepsilon_3}), \quad \Phi_0 \in C^{0,\alpha}_{2s-3}(\Pi_{\varepsilon_3}) \subset H^0_{\mu_1}(\Pi_{\varepsilon_3})$$
$$\Phi_1 \in C^{0,\alpha+1}_{2s-2}(T_{\varepsilon_3}) \subset H^{1/2}_{\mu_1}(T_{\varepsilon_3}), \quad \Phi_2 \in C^{0,\alpha+2}_{2s-1}(T_{\varepsilon_3}) \in H^{3/2}_{\mu_1}(T_{\varepsilon_3})$$

for all $\mu \in (2-s, 2) \subset (3/2, 2)$ and $\mu_1$ sufficiently close to $2-2s$, so that we can assume that $\mu_1 \in (1, 1+\varkappa)$. Thus we see that the conditions of Theorem 1 all hold, and this implies the validity of the formula (4.5) with $\eta \in H_{\mu_1}^4(\Pi_{\varepsilon_1})$, $\forall \varepsilon_1 > \varepsilon_3 > \varepsilon$. Lemma 2 now implies that the function $\eta$ has a bounded norm $\sup \rho^{\gamma-2}|\eta(x,\psi)|$, $(x,\psi) \in \Pi_{\varepsilon_1}$ where $\gamma = \mu_1 - 1 \in (0, \varkappa)$ is a number arbitrarily close to $\varkappa$. Now, since $\eta(x,\psi)$ satisfies the relations (4.1) with $\zeta_i = \Phi_i$, therefore $\eta \in C^{0,4+\alpha}_{2-\gamma}(\Pi_{\varepsilon_1})$, and this completes the proof of the theorem.

**7. Concluding remarks.** 1°. In Sect.1 we note the incompatability of the conditions of adhesion, conditions at the free boundary and the assumption $0 < \theta_m < \pi$ in the case of a plane stationary flow in a coordinate system attached to the moving point of contact. The arguments employed there can be transferred without major changes to the case of a nonstationary motion. We shall formulate the result obtained. Let the amount

$$E = \frac{\rho\nu}{2}\int_0^T\Big[\int_{\Omega(t)}\sum_{i,k=1}^{2}\Big[\Big[\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\Big)^2 dx_1\,dx_2\Big]\,dt$$

of kinetic energy of the fluid dissipated over the time $T$ be finite in the region of flow $\Omega(t)$, containing the moving point of contact at its boundary. Then the dynamic contact angle $\theta_m$ can assume one of the following two values: 0 or $\pi$, for all $t \in [0, T]$.

2°. Theorem 2 which gives the estimate for the residue term of (4.5), represents the sufficiently strict initial demands regarding the smoothness of the field of flow. The demands can be relaxed by considering the problem in terms of the physical variables $\mathbf{v}(x_1, x_2)$, $p(x_1, x_2)$ (this however makes the construction of the asymptotics more complicated). It is namely sufficient to require that $\Gamma$ be a Liapunov curve with index $\delta$ up to the point $O$, belonging everywhere, except at $O$, to the class $C^{3+\alpha}$, and, that the vector function $\mathbf{v}$ has a finite Dirichlet integral. Then by virtue of the lemma 3 of /24/ $\mathbf{v}$ satisfies the Hölder condition with the index $\delta_1 \leqslant \delta$ up to the point $O$ and belongs therefore (see /24/, Theorem 5) to the class $C_\delta^{0,2+\delta}$. Further smoothing of the field $\mathbf{v}$ is carried out according to a scheme analogous to that given above in the proof of Theorem 2.

3°. We shall give the asymptotic expressions for the tangential stress at the wall and the forms assumed by the free boundary near the point of contact. The equation of the free boundary is $x_2 = z(x_1, 0) \equiv f(x_1)$, $x_1 > 0$, since the image of this line in the $x\psi$-plane is a segment of the straight line $\psi = 0$. From (4.4) and (4.5) we obtain

$$f(x_1) = 4ka_1x_1^{2-\varkappa} + O(x_1^{2-\gamma}),\quad x_1 \to +0 \tag{7.1}$$

We see that the free boundary becomes smoother, the smaller the parameter $\varkappa = \pi^{-1}\,\mathrm{arctg}\,2k$ (or, with the remaining parameters fixed, the larger the coefficient of surface tension $\sigma$ entering the formula describing the capillary number $k = \rho\nu V/\sigma$). Since $\gamma > 0$ (although it can be arbitrarily small), we find that when $a_1 \neq 0$, then the curvature of the free boundary becomes infinite at the point of contact.

The dimensionless tangential stress at the wall equal to $2S_{12}(x_1, 0), x_1 < 0$ ($S$ is the deformation rate tensor) is written in the Mises variables in the form

$$2S_{12}|_{x_1<0,x_2=0} = [z_{\psi\psi}/(1 - z_\psi)^3]|_{x<0,\psi=0}$$

According to (4.4) and (4.5) we have

$$2S_{12}(x_1, 0) = 2a_0\,|x_1|^{-1/2} + O(|x_1|^{-\gamma}),\quad x_1 \to -0 \tag{7.2}$$

It is interesting to note that since $w_0(0) = 0$, the principal term of (4.5) makes no contribution towards the asymptotics of the free boundary (7.1). Conversely, the second term of (4.5) does not enter the asymptotics of tangential stress (7.2) since $w_1(\pi) = 0$.

4°. We shall comment briefly on the situation arising in the course of investigating the asymptotics near the point of contact when the liquid separates from the wall (problem of drying the capillary). The formal acceptance of the hypothesis $\theta_m = \pi$ leads in this case to the parameter $\varkappa$ becoming negative $-1/2 < \varkappa < 0$. At the same time the principal and the second term of the asymptotics in (4.5) must be interchanged, with $2 - \varkappa$ in the second term replaced by $1 - \varkappa$). The proof of Theorem 2 must also be changed somewhat. It is however physically unjustified to assume that when the liquid separates steadily from the wall, $\theta_m = \pi$. It could be postulated in analogy with the previous arguments that in this case $\theta_m = 0$. Computations however show that in this case a power asymptotics of the free boundary of the type (7.1) with the index greater than unity, and a power asymptotics of the velocity field continuous near the point of contact are both impossible (it is unclear whether the asymptotics could be superexponential). It seems that a scheme in which the points of contact are altogether absent and a liquid film remains on the surface of the capillary, with the thickness of the film asymptotically tending to a constant which may be equal to zero, is more realistic in the case of the stationary problem of drying a capillary.

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